**Existence, Uniqueness and Approximation**

**Differential Equation:** An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

**Example:** For examples of differential equation we list the following**:**

1. 

2. 

3. 

4. ****

**Classification:** Differential equations are classified on the basis of type as follows**:**

1. Ordinary Differential Equation (ODE),
2. Partial Differential Equation (PDE).

**Ordinary Differential Equation (ODE):** A differential equation involving derivatives of one or more dependent variables with respect to only one independent variable is called an ordinary differential equation.

**Example: 1. **

**2.**

**Partial Differential Equation (PDE):** A differential equation involving derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation.

**Example:** 1. 

2. 

**Order of a differential equation:** The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.

**Example:** 1.  is a first order differential equation.

2. **** is a second order differential equation**.**

**Degree of a differential equation:** The power of the highest ordered derivative involved in a differential equation is called the degree of the differential equation, after the equation is freed from radicals and fractions in its derivatives.

**Example:** 1.  is a differential equation of first degree.

2.  is a differential equation of second degree.

3. is a differential equation of first degree.

**Linear ordinary differential equation:** An ordinary differential equation of order n is called a linear ordinary differential equation of order n if it does not contain,

1. the transcendental functions of dependent variable,
2. the product of dependent variable and
3. the product of the derivatives of dependent variable.

It can be expressed as



where,  is not identically zero.

**Example**: 1. 

2. 

**Nonlinear ordinary differential equation:** A nonlinear ordinary differential equation is an ordinary differential equation that is not linear**.**

**Example:** 1. 

2. 

3. 

**Solution:** Any relation of dependent and independent variable which satisfies a differential equation is called a solution of that differential equation.

**General or Complete Solution:** If the solution of a differential equation of order  contains  arbitrary constants, then the solution is called a general solution of that differential equation.

**Example:** The general solution of the equation  is .

**Particular Solution:** If a solution is obtained from the general solution of a differential equation for definite value of arbitrary constant, then the solution is called a particular solution of the differential equation.

**Example:** The particular solution of the equation  is .

**Singular Solution:** A solution of differential equation which is not obtained from general solution for definite value of arbitrary constants and which is also not particular solution is called a singular solution.

**Example:** The general solution of  is  but it’s singular solution is .

**Initial Value Problem:** A differential equation with one or more supplementary conditions for same value of independent variable, which must satisfy the solution of the differential equation, is called an initial value problem.

**Example:** 1. 

2. 

**Boundary Value Problem:** A differential equation with more than one supplementary conditions for different values of independent variable, which must satisfy the solution of the differential equation, is called a boundary value problem.

**Example:** 1. .

**NOTE:** There are many general method for solving linear equations but no general method is known for solving nonlinear equations. The concept of general solution for linear equations differs from that for nonlinear equations. A first order linear equation has only one general solution, where as a nonlinear equation may have a general solution as well as singular solutions.

An initial value problem of first order may have

i). no solution

ii). one and only one solution

iii). more than one solution.

Now the question arises whether the initial value problem of first order has a solution. If yes, is the solution unique? The existence and uniqueness theorem gives the answer of the question.

**Lipschitz condition and Lipschitz constant:** A function is said to satisfy the Lipschitz condition in a rectangular region , where in *xy*-plane if there exists a positive constant such that

The constant is called a Lipschitz constant for the function .

**Question-01:** State the existence and uniqueness theorem for ,  in a bounded domain and prove the existence for it.

**Answer: Statement:** Let be a real- valued continuous function on the rectangle



where ,  and .

Further let  satisfies Lipschitz condition with Lipschitz constant  in .

i.e. 

where  is a constant independent of .

Then, the differential equation  has a unique solution  for which  .

**Proof:** We shall prove the theorem by method of successive approximations. Let  be such that . By successive approximations or Picard iterations we have



This proof consists of four parts:

**Step-1:** We prove that, for , the curve lies in the rectangle .

Now, 









This proves that the results holds for .

Assume that  lies in  and so  is defined and continuous and satisfies

 on .

Therefore from (4), we get











This shows that  lies in  and hence is defined and continuous on .

Thus by induction the result holds for all .

**Step-2:** We prove again by induction that



We have already verified (5) for  in first step where we have shown that

.

Assume that (5) holds for  in place of 

i.e. 

Then 













Hence by mathematical induction, we conclude that (5) is true for all .

**Step-3:** We shall prove that the sequence  converges uniformly to a limit for

.

Using the result (5), we have, the infinite series











Which is convergent for all values of  and . Consequently the above series is surely convergent.

**Step-4:** We now show that  satisfies the differential equation , that is ,

the existence of a solution of the given differential equation.

From (4) by taking limit we have





The integrand on the right hand side of (8) being a continuous function of , we conclude that the integral has the derivative.

Thus, the limit function satisfies the differential equation  on  and is such that .

**Question-02:** State the existence and uniqueness theorem for in a bounded domain and prove the uniqueness for it.

**Answer: Statement:** Let be a real- valued continuous function on the rectangle



where ,  and .

Further let  satisfies Lipschitz condition with Lipschitz constant  in .

i.e. 

where  is a constant independent of .

Then, the differential equation  has a unique solution for which  .

**2nd part**: Now we need to prove that the solution is the only solution for which .

By successive approximations or Picard iterations we have

+

If possible, let is another solution of the given problem.

Let , where

Then from (3) we have

Now from (3), (4) and (5) we have

Again by (4) we get

Now substituting (7) for the integrand in (6) we get

Again substituting (8) for the integrand in (6) we get

Continuing in this way, we shall surely get

Now the series converges, and so

Thus, can be made less than any number however small and consequently we conclude that

This shows that the solution is always unique.

**Question-03:** State and prove Cauchy- Peano existence theorem.

**Solution: Statement:** Let be a continuous function on

and ,

Then there exists a solution of on the interval such that

.

**Proof:** Consider a monotonically decreasing sequence of positive real numbers such that

as . Then for each , there exists an - approximate solution of

on the interval such that . If be one such solution for each , then

, where

This in turns imply that the sequence is equicontinuous on .

Putting in (2) we get,

.

This shows that the sequence is uniformly bounded by . Hence there exists a subsequence converges uniformly to a continuous function on

satisfying .

Now we define a function in the following way:

Then we have

Integrating (4) from to we get,

Also, since is an - approximate solution, so

Since uniformly on as , so from the uniform continuity of on we have uniformly on as . In fact as as uniformly on .

Replacing by in (5) we get,

Now putting we have

This complete the proof.

**Question-04:** Explain Picard’s method of successive approximation.

**Solution: Picard’s method:** Consider the following initial value problem

and

For Picard’s method of successive approximation, we choose a function as a zeroth approximation of the solution. Then a first approximation is determined in such a way that it satisfies the equations (1) and (2). i.e.

and

Now satisfies (3) and (4) if and only if

The second approximation , which satisfies (1) and (2) is as follows:

and

Now satisfies (6) and (7) if and only if

Similarly we can determine the third approximation , the fourth approximation and so on.

The nth approximation is determined as

where is the th approximation.

The exact solution of the initial value problem is

.

**Problem**

**Problem-01:** Find the Lipschitz constant for in Show that satisfies Lipschitz condition in .

**Solution:** The given function is , which is defined in the rectangular domain . Let , then

This shows that the function satisfies Lipschitz condition and the Lipschitz constant is .

**2nd part:** The given function is , which is defined in the rectangular domain

. Let , then

This shows that the function satisfies Lipschitz condition in .

**Problem-02:** Find the Lipschitz constant for , , . Is the solution of unique?

**Solution:** The given function is , which is defined in the rectangular domain .i.e.

Let , then

This shows that the function satisfies Lipschitz condition and the Lipschitz constant is .

**2nd part:** Here

It is evident that both and are continuous in the rectangular region

Hence the given equation has a unique solution.

**Problem-03:** Discuss the existence and uniqueness of a solution of the IVP , . Solve the IVP and the interval of existence.

**Solution:** Given initial value problem is

Comparing (1) with we get

and .

We put and then the rectangular region is

It is evident that is continuous in rectangular region .

Let , then

, where

Hence the function satisfies Lipschitz condition. Since is continuous and satisfies Lipschitz condition so the given initial value problem has a unique solution.

**2nd part:** We have

Using the initial condition we get

then (3) becomes

which is the required solution.

The solution is valid when

Thus the interval of the existence of the solution is

i.e. .

**Problem-04:** Discuss the existence and uniqueness of solution of the initial value problem

, and solve it.

**Solution:** Given initial value problem is

We know that the differential equation

with

has a unique solution through the point if both and are continuous in a rectangular domain

.

Comparing (1) with (2) we get,

and .

Here

It is evident that both and are continuous in domain containing the point. Hence the given equation has a unique solution with the initial condition.

**2nd part:** We have

Using the initial condition in (4), we get

then (4) becomes

which is the required solution.

**Problem-05:** Examine the uniqueness of solution of , and find the interval of existence.

**Solution:** Given initial value problem is

Comparing (1) with we get

and .

The rectangular region is

Here

It is evident that both and are continuous in domain .

Hence the given equation has a unique solution with the initial condition.

**2nd part:** Suppose, for and .

Then the existence and uniqueness theorem asserts that the given problem possesses a unique solution on .

Now

and so

Let

For maximum or minimum value of we know

When then

Therefore has a maximum value at and the maximum value is

Thus if then for all

and so

If then .

Thus, in any case

For , we have

The equality of gives

Thus, the given equation has a unique solution on the interval

.

**Problem-06:** Examine the existence and uniqueness of solution of the IVP , . Find also the solution and its region of validity.

**Solution:** Given initial value problem is

Comparing (1) with we get

and .

The rectangular region is

It is evident that is continuous in rectangular region .

Let , then

Since

and

Using these values in (3) we get

where

Hence the function satisfies Lipschitz condition. Since is continuous and satisfies Lipschitz condition so the given initial value problem has a unique solution.

**2nd part:** The equation (1) can be written as

This is a Bernoulli’s equation.

Dividing the equation (4) by we get

Put

Now the equation (5) becomes,

This is a linear equation.

Multiply both sides of equation (6) by we get

Integrating both sides of (7) we get

Using the initial condition in (8) we get

Putting the value of in (8) we get

**3rd part:** when and then is undefined.

Therefore the solution is valid in the interval

**Problem-07:** Show that the IVP , has a unique solution but the IVP

, has infinitely many solutions.

**Solution:** Given that ,

We know that the differential equation

with

has a unique solution through the point if both and are continuous in a rectangular domain

.

Comparing (1) with (2) we get,

and .

Here

It is evident that both and are continuous in domain containing the point

. Hence the given equation has a unique solution with the initial condition.

**2nd part:** Given that ,

Comparing (3) with (2) we get,

and .

Here

which is continuous in the whole plane but is continuous except .

Therefore the given equation has unique solution through any point with .

Also,

Integrating (4) we get

which has a continuous derivative for and is a solution through for every value of . Thus the IVP , has infinitely many solutions.

**Question-08:** Using Picard’s method of successive approximation find the first three approximations to the solution of

**Solution:** Given initial value problem is

Comparing (1) with we get

and .

By picard’s method of successive approximation the zero approximation of the actual solution is

.

The 1st approximation is

The 2nd approximation is

The 3rd approximation is

**Question09:** Use Picard’s method of successive approximations to find the solution of the IVP

, and verify your result.

**Solution:** Given initial value problem is ,

Comparing (1) with we get

and .

By picard’s method of successive approximations the zero approximation of the actual solution is

.

The 1st approximation is

The 2nd approximation is

The 3rd approximation is

Proceeding in this way we have

Therefore the required solution is,

**2nd part: Verification:** The solution is

Differentiating (2) w.r.to ‘t’ we get

From (2) and (3) we get

Also putting in (2) we get

Thus, the solution satisfies the given differential equation with the given initial condition.

**Question-10:** Use Picard’s method of successive approximations to find the third approximate solution of

**Solution:** Given initial value problem is

Comparing (1) with we get

and .

By picard’s method of successive approximation the zero approximation of the actual solution is

.

The 1st approximation is

The 2nd approximation is

The 3rd approximation is

This is the third approximate solution.